

**FINAL REPORT**

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**DARPA**

**Quantum Information Science and Technology (QuIST)**

**for a program of research entitled:**

**SOLID-STATE NMR QUANTUM COMPUTER**

**for the period of**

**SEPTEMBER 30, 2000 – SEPTEMBER 29, 2005**

**AFOSR Grant F49620-01-1-0556**

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## I. Publications

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- L. D. Ladd, Y. Yamamoto, J. R. Goldman, and F. Yamaguchi, "Solid State Crystal Lattice NMR Quantum Computation", *Quantum Information and Computation*, 1, Special Issue, 56-81 (Dec 2001)
- T. D. Ladd, J. R. Goldman, F. Yamaguchi, Y. Yamamoto, E. Abe, and K. Itoh, "All Silicon Quantum Computer", *Phys. Rev. Lett.* **89**, 017901-1 - 017901-4 (July, 2002)
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- C. P. Master, F. Yamaguchi, Y. Yamamoto, "Efficiency of Free-Energy Calculations of Spin Lattices by Spectral Quantum Algorithms," *Phys. Rev. A* **67**, 032311-1 - 032311-9 (March 2003).
- A. S. Verhulst, D. Maryenko, Y. Yamamoto, K. M. Itoh, "Double and Single Peaks in Nuclear Magnetic Resonance Spectra of Natural and  $^{29}\text{Si}$  Enriched Single Crystal Silicon," *Phys. Rev. B* **68**, 054105-1 - 054105-6 (August 2003).
- K. C. Fu, T. D. Ladd, C. Santori, Y. Yamamoto, "Optical Detection of the Spin State of a Single Nucleus in Silicon," *Phys. Rev. B* **69**, 125306-1 - 125306-5 (March 2004).
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- T. D. Ladd, D. Maryenko, Y. Yamamoto, E. Abe, K.M. Itoh, "Coherence time of decoupled nuclear spins in silicon," *Phys. Rev. B* **71**, 014401-1 - 014401-12 (January 2005).

- A. S. Verhulst, Y. Yamamoto, K. Ito, "Optical Pumping of  $^{29}\text{Si}$  Nuclear Spins in Bulk Silicon at High Magnetic Field and Liquid Helium Temperature," *Phys. Rev. B* **71**, 235206-1 - 235206-10 (June 2005).
- K.C. Fu, C. Santori, C. Stanley, M.C. Holland, Y. Yamamoto, "Coherent Population Trapping of Electron Spins in a High-Purity n-Type GaAs Semiconductor," *Phys. Rev. Lett.* **95**, 187405-1 - 187405-4 (October 2005).

Presented at conferences:

- Y. Yamamoto, "Solid State Crystal Lattice NMR Quantum Computer," DARPA Quantum Information Science and Technology (QuIST) Kickoff Meeting, Dallas, Texas (November 26-29, 2001)
- T. D. Ladd, J. R. Goldman, A. Verhulst, F. Yamaguchi, Y. Yamamoto, E. Abe and K. M. Itoh, "Crystal Lattice Quantum Computation," DARPA Quantum Information Science and Technology (QuIST) Kickoff Meeting, Dallas, Texas (November 26-29, 2001)
- Y. Yamamoto, T. Ladd, J. R. Goldman, A. Verhulst, C. P. Master and F. Yamaguchi, "Solid state NMR quantum computation," 2002 Winter Conference on Condensed Matter Physics *Quantum Coherence and Dissipation*, Aspen, Colorado (February 10-16, 2002)
- Y. Yamamoto, T. Ladd, J. R. Goldman, A. Verhulst, C. P. Master, F. Yamaguchi, K. M. Itoh and E. Abe, "Crystal lattice NMR quantum computation," International Symposium on quantum Computing, Tokyo Japan (March 12-14, 2002)
- J. R. Goldman, T. D. Ladd, A. S. Verhulst, A. Dana, F. Yamaguchi, Y. Yamamoto, E. Abe and K. Itoh, "An All-Silicon Quantum Computer Using Magnetic Resonance Force Microscopy," 1<sup>st</sup> International Conference and School, Nanoscale/Molecular Mechanics, Maui, Hawaii (May 12-17, 2002)
- A. S. Verhulst, J. P. Strahan and Y. Yamamoto, "Optical pumping of  $^{28}\text{Si}$  and  $^{29}\text{Si}$  for State Initialization of an All-Optical Silicon Quantum Computer," The 6<sup>th</sup> International Conference on Quantum Communication and Computing (QCMC'02), Cambridge, Massachusetts (July 22-26, 2002)
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- F. Yamaguchi, "Quantum Simulation of High Temperature Superconducting Oxides," The 8th International Symposium on Advanced Physical Fields (APF8): Advanced Materials for Quantum Computing, Tsukuba, Japan (January 14-17, 2003).
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- Y. Yamamoto, "Photonic Qubit and Nuclear Qubit for Quantum Information Systems," ERATO Conference on Quantum Information Science 2003 (EQIS '03), Kyoto, Japan (September 4-6, 2003).
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- Y. Yamamoto, "Progress Toward All Silicon Quantum Computers," QuIST (Quantum Information Science and Technology) Fall Program Review, Ft. Lauderdale, Florida (November 12-14, 2003).
- F. Yamaguchi, C. P. Master, Y. Yamamoto, N. Khaneja, "Simulation of Many-Body Interaction and Multiqubit Logic Gates by Ising Spin Chains," QuIST (Quantum Information Science and Technology) Fall Program Review, Ft. Lauderdale, Florida (November 12-14, 2003).
- F. Yamaguchi, "Solid-State NMR Quantum Computation," 3rd Japan-American Frontiers of Engineering (JAFoE) Symposium, Irvine, California (November 20-22, 2003).
- F. Yamaguchi, T. D. Ladd, A. S. Verhulst, J. R. Goldman, Y. Yamamoto, "All-Silicon Quantum Computer," Lorents Center Workshop: Fundamentals of Solid State Quantum Information Processing, Leiden, The Netherlands (December 8-12, 2003).
- A. S. Verhulst, Y. Yamamoto, K. Itoh, "Enhancement of the  $^{29}\text{Si}$  Polarization in Bulk Silicon at High Magnetic Field (7T) and Low Temperature (4K) via Optical Pumping," The 45th ENC (Experimental Nuclear Magnetic Resonance Conference, Pacific Grove, California (April 18-23, 2004).

- F. Yamaguchi, T. D. Ladd, J. R. Goldman, A. S. Verhulst, C. P. Master, K. Fu, C. Santori, S. Koseki, Y. Yamamoto, "Solid-State Crystal Lattice NMR Quantum Computer," DARPA Quantum Information Science and Technology (QuIST) Review Meeting and Workshop - Scaleablity/Error Control, Chicago, Illinois (May 4-7, 2004).
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- J. Goldman, T. Ladd, C. Santori, S. Koseki, G. Solomon, B. Zhang, Y. Matsumoto, F. Yamaguchi, Y. Yamamoto, "Large Magnetic Field Gradients for Crystal Lattice Quantum Computing," American Physical Society (APS) March Meeting 2005}, Los Angeles, California (March 25, 2005).
- Y. Yamamoto, "Nuclear spin quantum memory in semiconductors," DARPA Quantum Information Science and Technology (QuIST) Program Review, Augustine, Florida (April 5-7, 2005).

## **II. Report of inventions / Technology transfer / Copies of technical reports**

US patent:

- 6,437,413, "Crystal lattice quantum computer," August 2002, Inventors: F. Yamaguchi, Y. Yamamoto

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#### IV. Scientific progress and accomplishments

We have obtained experimental evidence confirming each component of our proposed scheme for nuclear spin-based large scale quantum computation. Long decoherence time, individual accessibility and optical initialization/readout of nuclear spins are promising. A semiconductor EIT system shows promise as an interface between the nuclear spins and photonic qubits. Deterministic, indistinguishable single-photon sources from the use as photonic qubits have been confirmed in our group and others. Our accomplishments over the period are described in detail below.

The remaining challenge in the future is to combine all of these elements in a single system. Our first step would be to combine them for small-scale quantum computation, such as a quantum repeater necessary for long-distance quantum communication. Then, the final goal is to improve each component for large-scale quantum computation based on multiple quantum information processors made of nuclear spins and quantum communication among them by photonic qubits.

##### 1. Material and coherence time experiments

We have completed coherence time measurement of  $^{29}\text{Si}$  nuclear spins at room temperature. Natural silicon consists of 95.33% of  $^{28}\text{Si}$  and  $^{30}\text{Si}$ , which both have spin 0, and 4.67% of  $^{29}\text{Si}$ , which has spin  $\frac{1}{2}$ .

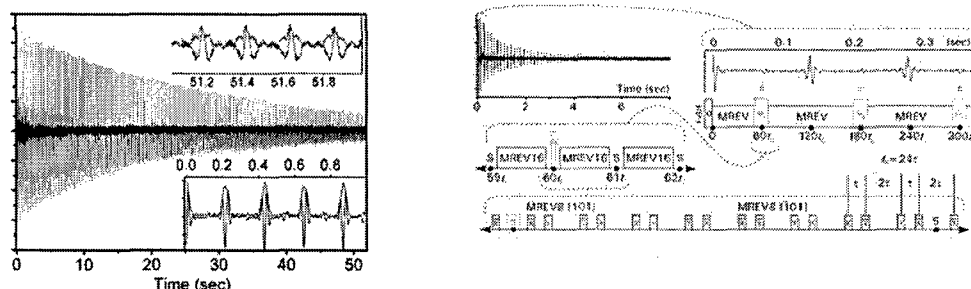


Figure 1: The spin echo data (left) from an isotopically natural single crystal of silicon under CPMG-MREV-16 pulse sequence (right).

The measured decoherence times are  $T_1 = 272$  minutes, and  $T_2 = 25$  seconds.  $T_1$  corresponds to the rate of energy exchange with the environment, and  $T_2$  to the loss of phase coherence. The relevant timescale for quantum information is the latter. This corresponds to a quality factor  $Q = 10^9$ , comparable to the highest value  $Q = 10^9$  of liquid state NMR in other proposed quantum computation methods. This long decoherence time is the largest advantage of silicon. In our measurement, the decoherence time is that of isolated  $^{29}\text{Si}$  nuclei. Decoherence due to couplings among the nuclear spins is eliminated by applying NMR pulse sequences. The spin echo data from the nuclear spins and the decoupling pulse sequences are shown in Figure 1.

## 2. Micromagnet design

We have designed a micro-magnet to create a magnetic field gradient as shown in Figure 2. When it is made of a ferromagnetic material, Dysprosium, our simulated result shows that the magnetic field gradient created by the magnet is as large as  $10 \text{ Tesla}/\mu\text{m}$  in one direction and  $1 \text{ Tesla}/\mu\text{m}$  in a perpendicular direction, in the presence of a large magnetic field.

In this configuration, a crystalline solid used for quantum computation will be embedded in the micro-magnet. The magnetic field gradients created by a micro-magnet introduce

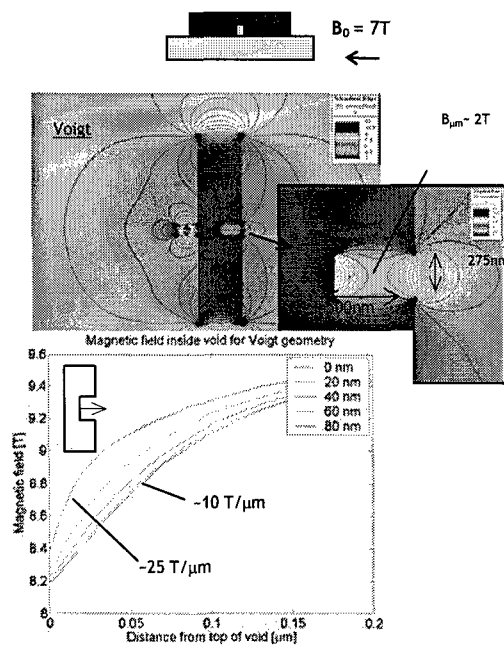
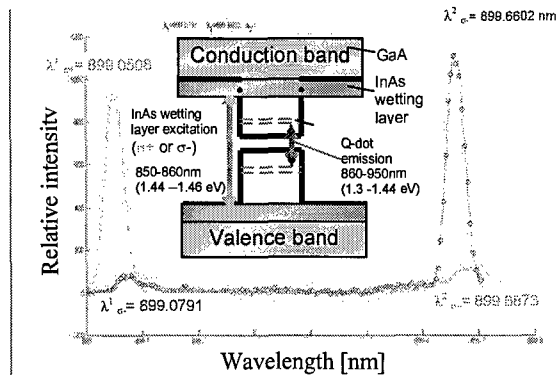


Figure 2 A simulated magnetic field created by a Dysprosium micro-magnet





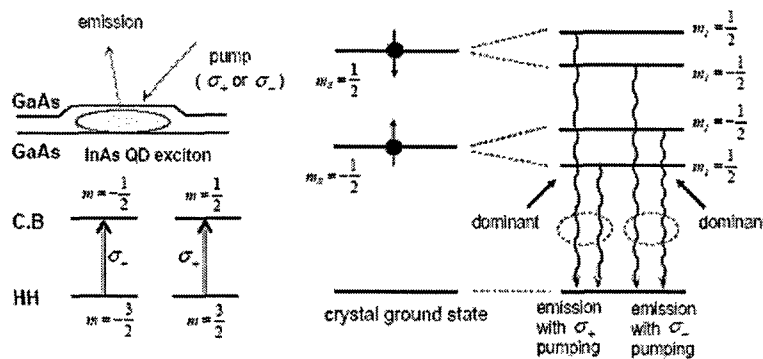
**Figure 3: Overhauser shift in InAs quantum dots due to optical pumping of nuclear spins. Nuclear polarization corresponds to  $60\% \pm 10\%$  for  $\sigma_+/\sigma_-$  polarized light.**

gradients in nuclear Larmor frequencies in the crystalline solid. It will allow individual access of nuclear spins by choosing corresponding frequencies.

We have characterized micro-magnets by Zeeman splitting, proportional to a magnetic field, in photoluminescence (PL) from InAs quantum dots. The results of Zeeman splitting vs. dc field are in agreements with theory and show that the local magnetic field provided by a micro-magnet is about 1T anti-parallel to the external magnetic field.

### 3. Optical pumping

Our proposed scheme employs optical pumping as a method of polarizing the nuclear spins in a crystalline solid at the beginning of a quantum computation. Electrons near nuclear spins are optically excited in a form of a free exciton or bound exciton state.



**Figure 4: Level structure and selection rule of an InAs quantum dot.**

Conventionally spin-polarized electrons in direct band-gap semiconducting materials, such as GaAs and InAs, have been created by circularly polarized laser light by the selection rule. The nuclear spins are coupled with the electron spins via the hyperfine interaction, which causes the nuclear spins and the electron spins to flip simultaneously. Constant illumination of the materials with a circularly polarized light favorably polarizes the nuclear spins in a single direction.

Figure 3 shows the Overhauser shift in InAs quantum dots due to optical pumping of nuclear spins. When pumped by  $\sigma_+$  polarized light, spin  $+\frac{1}{2}$  electrons are excited in a quantum dot, leading to more population in the nuclear ground state (spin  $+\frac{1}{2}$ ). Consequently, the electron spins' Zeeman splitting increases. When pumped by  $\sigma_-$  polarized light, the Zeeman splitting decreases. (See Figure 4.)

We have completed optical pumping of nuclear spins in silicon. Polarization of nuclear spins in silicon by polarized light does not work because silicon is an indirect band-gap material. The circular polarization of the light does not transfer to the spin polarization of electron spins, and the polarization mechanism is inefficient. Nuclear spin polarization by optical pumping that had been observed was less than 0.01% at 77 Kelvin in a low magnetic field (1 to 100 Gauss) because of this inefficiency.

Electron spins cannot be polarized by circularly polarized laser light, but they can be polarized at low temperature in a high field due to the Boltzmann distribution. Unpolarized light destroys the electron spin polarization, equally populating the up-spin states and down-spin states. The electron system equilibrates towards its ground-state spin state by flipping nuclear spins simultaneously.

We have achieved for the first time 0.3% nuclear polarization in silicon at 4 Kelvin in a high magnetic field (7 Tesla). We expect to reach 10% in a magnetic field of 11 Tesla.

Figure 5 shows the observed nuclear polarization in silicon at various laser powers. Successive application of  $\pi/2$  pulses destroys the nuclear polarization at the beginning of each measurement. When excited below the bandgap  $E_0$ , there is no optical pumping effect and nuclear polarization towards equilibrium distribution builds up. (The population of the ground state of nuclear spins increases in time, starting at zero.) When excited above the bandgap  $E_0$ , the effect of optical pumping is observed, leading to a negative population of nuclear spins. (The excited state is more populated than the ground state.)

#### 4. Optical detection of nuclear spins

We propose to detect the state of nuclear spins at the end of quantum computation using nearby electrons. In semiconductors, free excitons are trapped at impurity sites (donors or acceptors), forming bound excitons. A bound exciton has relatively narrow linewidth due to its localized nature. If the hyperfine interaction between the impurity nucleus and the impurity bound exciton is sufficiently strong, the bound exciton photoluminescence energy detects the nuclear spin state.

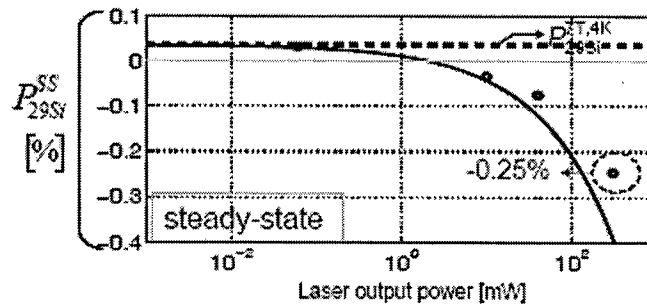
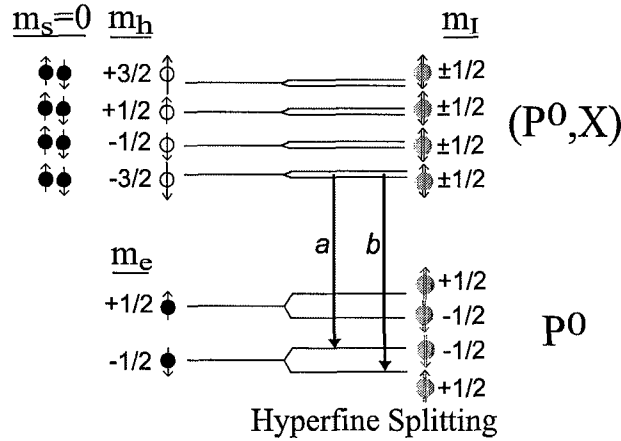


Figure 5: Steady state polarization of  $^{29}\text{Si}$  nuclear spins under laser irradiation at 7T and 4.2K. Solid line represents theoretical prediction and the dashed line represents thermal equilibrium polarization in the absence of laser irradiation.



**Figure 6** Energy diagram for the neutral donor  $P^0$  and its donor bound exciton  $(P^0, X)$  in a magnetic field.

Figure 6 shows an energy diagram for a natural donor ( $P^0$ ) and its donor bound exciton ( $P^0, X$ ) in a magnetic field in the particular case of a  $^{31}\text{P}$  donor in a  $^{28}\text{Si}$  matrix. The bound exciton energy is split into four hole-Zeeman levels. The hyperfine interaction between a nucleus and a hole is negligibly small compared with the one between an electron and a nucleus due to the symmetry of their wavefunctions. The nuclear spin state determines the bound exciton photoluminescence energy.

Our theoretical estimate indicates a likelihood of single nuclear detection by this method. The donor ground state is split due to the hyperfine coupling to the nuclear spin by 60MHz. PL linewidth of a sing  $^{31}\text{P}$  donor impurity would be 3MHz, which is much smaller than the hyperfine splitting of 60MHz. The  $(P^0, X)$  state decays primarily via a non-radiative Auger process with a lifetime of 300ns. However, a zero-phonon radiative channel with a lifetime of 2ms limits photon flux. The signal-to-noise ratio for direct frequency discrimination of PL by Mach-Zender interferometer reaches unity, assuming 10MHz PL linewidth, 10% collection efficiency of a single photon counter and 0.1 second integration time.

##### 5. Photonic qubit and nuclear qubit for quantum information systems

Quantum teleportation network must inevitably be utilized in large-scale quantum computation. This is mainly because the use of only nearest-neighbor couplings between qubits demands too large an overhead in logic operations on remote qubits, and quantum computers with more than about  $10^3$  qubits are hard to achieve in practice. A quantum teleportation network for quantum computation comprises the following procedure.

- Capturing and/or storing qubit information in quantum memories in independent quantum information processors,

- Entanglement purification and swapping performed in the processors; and
- Simple quantum computation such as quantum error correction performed in the processors.

For this procedure to succeed, the decoherence time  $T_2$  of the quantum memories must be longer than the communication delay between remote memories. The only physical systems that enjoy such a long decoherence time are nuclear spins, due to their good isolation from the environment. Moreover, a spin-1/2 nucleus intrinsically provides a two-level structure for a qubit. Precise control of nuclear spins by well-developed NMR techniques is an additional benefit.

Formation of non-local entanglement would be possible only via photons, which do not directly interact with nuclear spins. Electron spins in a semiconductor interface nuclear spins with photons via the hyperfine interaction. Examples of electron systems that have long enough  $T_2$  time for transferring photonic qubit information to nuclear spins or vice versa include a trapped electron at an N-V center in diamond. The decoherence time  $T_2$  of that system has been measured 6 – 60  $\mu\text{sec}$  at room temperature.

Specifically, our suggested scheme for quantum computation includes three components: (1) A photonic qubit network to create initial EPR-Bell states in remote quantum information processors; (2) Quantum information processors with quantum memories made of nuclear spins in a semiconductor to store and purify the EPR-Bell states; and (3) An interface between photonic qubits and nuclear spin qubits made of electron spins.

Semiconducting materials provide a good interface between photonic qubits and nuclear spin qubits by means of an intermediate electron spin. In particular, a bound exciton in a semiconductor in a magnetic field offers capability for electromagnetically induced transparency (EIT), allowing trapping and storing photonic qubit information. In such a system, double degeneracy of the ground state is lifted by Zeeman shift in the presence of a magnetic field. Those two states and one of the bound exciton states, appropriately chosen for desired transitions to the ground states, comprise our EIT system.

Specifically, we study an ensemble of donors in GaAs bulk crystal, in which the level structures are as shown in Figure 7. When trapping a photon in a probe pulse, we turn on a coupling pulse that connects the excited electron Zeeman state and one of the bound exciton states. The probe pulse is trapped by adiabatically turning off the control pulse after the photon enters in the EIT device. The reverse process releases the photon from it.

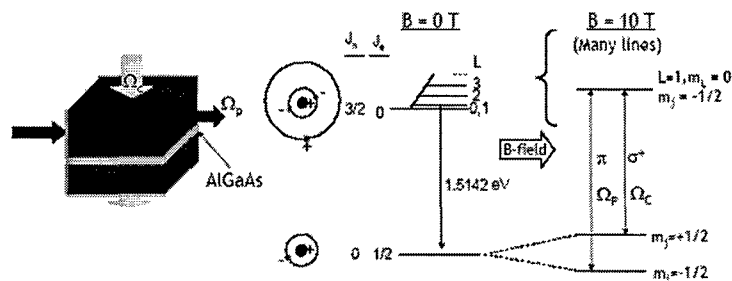


Figure 7: Level structures (ground states and bound exciton states) of GaAs/AlGaAs EIT system.

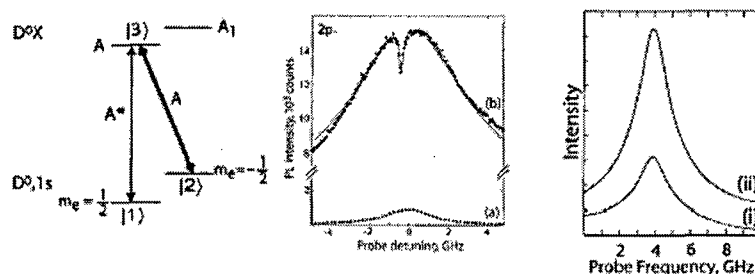


Figure 8: Observation of EIT in GaAs

Figure 8 shows observation of such phenomena in a sample consisting of 10  $\mu\text{m}$  GaAs and 4  $\mu\text{m}$   $\text{Al}_{0.3}\text{Ga}_{0.7}\text{As}$  layers. The PL experiment was performed at 1.5 Kelvin in a magnetic field of 7T. The results are in good agreement with a theoretical steady-state three-level density matrix model.

#### 6. Efficient decoupling and recoupling in solid state NMR

We have developed a scheme for decoupling and selectively recoupling large networks of dipolar-coupled spins. The scheme relies on a combination of broadband, decoupling pulse sequences applied to all the nuclear spins with a band-selective pulse sequence for single spin rotations or recoupling. The evolution-time overhead required for selective coupling is independent of the number of spins, subject to time-scale constraints, for which we discuss the feasibility. This scheme may improve the scalability of solid-state-NMR quantum computing architectures.

#### 7. Quantum algorithm for free-energy calculation of spin lattices by spectral quantum algorithm

Ensemble quantum algorithms are well suited to calculate estimates of the energy spectra for spin-lattice systems like ours. Based on the phase estimation algorithm, these algorithms

efficiently estimate discrete Fourier coefficients of the density of states. Their efficiency in calculating the free energy per spin of general spin lattices to bounded error is examined. We found that the number of Fourier components required to bound the error in the free energy due to the broadening of the density of states scales polynomially with the number of spins in the lattice. However, the precision with which the Fourier components must be calculated is found to be an exponential function of the system size.